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By

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and

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ABSTRACT

We examine a linear fractional flow model which can be interpreted as a Markov chain with partially controlled transition probabilities. The paper classifies the set L of limiting distributions and details several of its properties. A precise classification in a three dimensional case is presented.

1. INTRODUCTION

This paper identifies the set of limiting solutions for the n dimensional constrained linear system

$$
\begin{gathered}
x(t+1) = x(t)P + u(t) , \\
x(t)e = 1 , \; u(t) \geq 0 \qquad t = 0,1,2, \ldots
\end{gathered} \tag{1}
$$

where e is a column vector with each of its n elements equal to one. The initial vector $x(0) \geq 0$ is given, and the $n \times n$ matrix P is nonnegative. In general, we assume $w = (I - P)e \geqq 0$ and that $(I - P)$ has an inverse. Our principal result, however, requires slightly stronger assumption. It follows that $x(t)$ is always a nonnegative vector with components summing to one; i.e., $x(t)$ is the distribution of some quantity at time t. Equation (1) shows how that distribution can change over discrete time.

Bartholomew, [1] and [2], has derived an equivalent expression of the dynamics (1) in which $x(t)$ is the distribution of a partially controllable Markov process. The equivalence is based on the identity $x(t)w = u(t)e$, which holds if $x(t)$ and $u(t)$ solve (1). For any solution of (1) we define $z(t)$ and $Q[z(t)]$ by

$$
z(t) = \begin{Bmatrix} x(t) & \text{if} \quad u(t) = 0 \\ u(t)/u(t)e & \text{otherwise} \end{Bmatrix} , \tag{2}
$$

and

$$
Q_{ij}[z(t)] = P_{ij} + w_i z_j(t) ,
$$

or in matrix notation[†]

$$
Q[z(t)] = P + wz(t) .
$$

[†]Since w is a column vector, $wz(t)$ is an $n \times n$ matrix.

Note that $z(t) \geq 0$, $z(t)e = 1$, and that $Q_{ij}[z(t)]$ is a stochastic matrix. It follows that

$$x(t+1) = x(t)Q[z(t)] \ . \tag{3}$$

Now, in contrast, suppose $z(t)$ is any sequence with $z(t) \geq 0$, $z(t)e = 1$. Given $x(0)$, we define $Q[z(t)]$ and $x(t)$ by (2) and (3). It is apparent that $u(t) = [x(t)w]z(t)$ and $x(t)$ will solve (1).

This paper characterizes the set L of limiting distributions. We can say, roughly, that for any $x(t)$ and $u(t)$ satisfying (1), $x(t)$ converges geometrically to L . The set L has two other interesting properties. First, let a closed set A be defined as a *trapping set* if $x(0) \in A \Rightarrow x(1) \in A$. We find that L is the smallest trapping set. Second, if $x(0) \notin L$, then it is not possible to return to $x(0)$; in contrast, if $x(0)$ is in the relative interior of L , it is possible to return to $x(0)$ in a finite number of periods.

Section 2 motivates system (1) in the context of a manpower planning problem. Section 3 is devoted to definitions, a statement of the theorem, and a discussion of the result. In Section 4 we examine a special case with $n = 3$, and obtain a precise characterization of the set L . Proofs are included in Section 5.

This paper extends and strengthens several results of Toole [7]. Specific references to Toole's work is included as it appears. For completeness we have included short proofs of several of Toole's results.

2. MOTIVATION-MANPOWER FLOW

Consider an organization with n job classifications called ranks. Let M_{ij} be the fraction of workers in rank i that move to rank j in one period and let $v_j(t) \geq 0$ be the number of workers hired into rank j in period t. Finally let $y_j(t)$ be the number in rank j at time t. It follows that

$$y_j(t+1) = \sum_{i=1}^{n} y_i(t)M_{ij} + v_j(t) , \tag{4}$$

or in matrix notation $y(t+1) = y(t)M + v(t)$.

The initial inventory of manpower is given by $y(0) \geq 0$. Assume the organization is growing constantly at rate $(\theta - 1)$; thus the size at time t is $\theta^t y(0)e$. Let $x(t)$ be defined as $x(t) = y(t)/\theta^t y(0)e$, and define $u(t) = v(t)/\theta^{t+1} y(0)e$. Then $x(t)$ and $u(t)$ obey Equation (1) of Section 1 with $P = M/\theta$.

As a second manpower example consider an organization with $n - 1$ ranks. Define y , v , and M_{ij} as above. We add rank n to the organization to consist of unfilled positions, and let $y_n(t)$ denote the number of unfilled positions at time t . Define $v_n(t)$ as the number of positions open during t which remain open in the next period, set $M_{ni} = 0$ for $i = 1,2, \ldots, n$ and $M_{in} = 1 - \sum_{j=1}^{n-1} M_{ij}$ for $i = 1,2, \ldots, n-1$. With these definitions Equation (4) holds for our second manpower system.

3. DEFINITIONS AND MAIN RESULT

We define the norm of a vector $x \in \mathbb{R}^n$ to be $||x|| = \sum_{j=1}^{n} |x_j|$. The closure, relative interior, and convex hull of sets are denoted cl , ri , and C respectively. We let ϕ denote the empty set. The simplex $S = \{x \mid xe = 1 , x \geq 0\}$ is the set of possible distributions and we topologize the closed subsets of the metric space $(S, || \; ||)$ with the Hausdorf metric, [3]:

(5)
$$\delta(A,D) = \underset{x \in A}{\text{Max}} \; \underset{y \in D}{\text{Min}} \; ||x - y||$$
$$d(A,D) = \text{Max} \; [\delta(A,D), \delta(D,A)] \; .$$

Let $E = \{x \mid x \in S, x \geq xP\}$ be defined as the set of equilibrium distributions; then the solution $x(t) = x$ for all t is feasible for (1) if and only if $x \in E$. Now choose any $z \in S$ and consider the stochastic matrix $Q(z) = P + wz$ where $w = (I - P)e \geqq 0$. When $(I - P)$ has an inverse N , Toole [7] has demonstrated that $x = zN/zNe$ is the unique $x \in S$ such that $x = xQ(z)$. It follows that $x \in E$ if and only if x is a stationary vector of some stochastic matrix $Q(z)$ for $z \in S$.

Suppose $x = xQ(z)$, $x \in S$. It does *not* follow that for any $x(0)$, with $u(t) = (x(t)w)z$, we have $x(t) \to x$. Consider P and z below.

(6)
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad z = (1,0,0)$$
$$Q[z] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow x = (1/3,1/3,1/3)$$

However if $x(0) = (\alpha,\beta,\gamma)$, then $x(1) = (\gamma,\alpha,\beta)$, $x(2) = (\beta,\gamma,\alpha)$, $x(3) = (\alpha,\beta,\gamma)$. When z is strictly positive, the Markov matrix $Q(z)$ is regular and we then have $x(t) \to x$ for any initial $x(0)$.

Let A be any nonempty subset of S and $R(A) \stackrel{\Delta}{=} \{x \mid x \in S, x \geq yP, y \in A\}$.

When A is a singleton, $\{x\}$, we use the notation $R(x)$. For $A \neq \phi$, define $R^0(A) = A$, $R^1(A) = R(A)$, and for $t \geq 1$, $R^{t+1}(A) = R(R^t(A))$. For any $x \in S$, $R^t(x)$ is the set of $x(t)$ feasible in (1) given that $x(0) = x$.

It is easy to verify that if A is closed, convex, or polyhedral then $R(A)$ will have the same property. Moreover, $A \subseteq B \Rightarrow R(A) \subseteq R(B)$, and $CR(A) = R(CA)$, and $\mathrm{cl}\, R(A) = R(\mathrm{cl}\, A)$. We also can see that for any t, $E \subseteq R^t(E) \subseteq R^t(S) \subseteq S$. Therefore we define the limiting set L as $L = \bigcap_{t=0}^{\infty} R^t(S)$. Note that L is nonempty, closed and convex. Toole [7] has demonstrated

Proposition 1:

$$R(L) = L .$$

Proof:

$y \in R(L) \Rightarrow y \in S$ and $y \geq xP$ for some $x \in L \subseteq R^t(S)$ for all $t \geq 0$. Thus $y \in R^{t+1}(S)$ for all $t \geq 0 \Rightarrow y \in L$. Conversely if $x \in L$, then $x \in S$, and since $x \in R^t(S)$ for all $t \geq 1$, there exists a $y(t) \in R^{t-1}(S)$ such that $x \geq y(t)P$. Let y be an accumulation point of the $y(t)$. It follows that $y \in L$, thus $x \geq yP \Rightarrow x \in R(L)$. ■

Proposition 2: (Stanford [6], Toole [7])

If $(I - P)$ has an inverse then

(i) $\mathrm{int}\, E \neq \phi$.

(ii) For any $x(0)$, and $y \in \mathrm{int}\, E$, there exists a finite T such that $y \in R^t(x(0))$ for $t \geq T$.

Proof:

We have $(I - P)^{-1} \geq 0$. If $b > 0$, then $y = b(I - P)^{-1} > 0$, and $x = y/ye$ satisfies $x \in S$, $x > xP$.

With x defined as above, we have $z = \frac{x(I - P)}{xw}$, a strictly positive appointment vector. Therefore $x(0)Q^t(z) \to x \in \text{ri } E$. There exists an ε neighborhood $N_\varepsilon(x)$ of x such that for any $y \in N_\varepsilon(x)$ we have $x \in R(y)$, or $y \geq xP$. There is a finite T such that $x(0)Q^{T-1}(z) \in N_\varepsilon(x)$. Thus $x \in R^T(x(0))$, and since $x \in E$, we have $x \in R^t(x(0))$ for all $t \geq T$.■

For $k \geq 1$, define the k^{th} cycle set as $C^k = \{x \mid x \in R^k(x)\}$. If $x \in C^k$, then it is possible to return to x in k steps; note that $C^1 = E$. Define $C^\infty = \bigcup_{k=1}^{\infty} C^k$; if $x \in C^\infty$, there is some finite k such that it is possible to return to x in k steps.

<u>Proposition 3</u>: (Toole [7])

$$\text{cl } C^\infty = \text{cl}\left[\bigcup_{t=1}^{\infty} R^t(E)\right] \subseteq L$$

Our theorem strengthens this result considerably.

<u>Theorem 1</u>:

If $w = (I - P)e > 0$, then

(i) L is the unique closed nonempty subset of S satisfying $R(L) = L$.

(ii) For any closed nonempty subset A of S ,

$$d(R^t(A),L) \leq \sigma^t d(A,L)$$

where $0 \leq \sigma < 1$.

(iii) $\text{cl } C^\infty = L$

(iv) If $\phi \neq A \subseteq S$, then $R(A) = A$ implies $\text{cl } A = L$.

The proof will be presented in Section 5. It depends in large part on the following lemma:

<u>Lemma</u>:

Let $\sigma = [1 - 1/2 \text{ Min } \{w_i \mid i = 1,2, \ldots, n\}] < 1$.

For any x , y , $z \in S$,

$$||(x - y)Q(z)|| \leq \sigma||x - y|| .$$

We suspect that a weaker version of the theorem is valid under the weaker hypothesis that $(I - P)$ is nonsingular. However, the lemma does not generalize with that hypothesis. Recall example (6). Let $z = (1,0,0)$, $y = (0,1,0)$ and $x = (0,0,1)$. We have

	$xQ^t(z)$	$yQ^t(z)$	$\|\|(x - y)Q^t(z)\|\|$
$t = 0$	(0,0,1)	(0,1,0)	2
$t = 1$	(1,0,0)	(0,0,1)	2
$t = 2$	(0,1,0)	(1,0,0)	2
$t = 3$	(0,0,1)	(0,1,0)	2
⋮	⋮	⋮	⋮

This example has

$$||(x - y)Q^t(z)|| = ||x - y|| \quad \text{for all} \quad t .$$

Before concluding this section we shall discuss several implications of the theorem. First $R^t(A) \to L$ for any nonempty subset A of S . For any A we have $R^t(S) \supseteq R^t(A)$. In addition, there exists a finite T and $x \in riE$ such that $x \in R^T(A)$ thus $R^t(A) \supseteq R^{t-T}(x)$ for all $t \geq T$. The sequences $R^t(S)$ and $R^{t-T}(x)$ are closed and converge geometrically to L . Moreover, the sequence $R^t(S)$ is contracting, and the sequence $R^{t-T}(x)$ is expanding.

As a second point, let A be any closed subset such that $R(A) \subseteq A$. To see that $L \subseteq A$, assume $x \in L$ and $x \notin A$. It follows that $x \notin R^t(A)$ for all t , and therefore that

$$d[R^t(A),L] \geq d[A,L] \geq \min_{y \in A} ||x - y|| > 0 .$$

However, the term on the left converges geometrically to zero.

We observe if $x \notin L$, then it is impossible to return to x. If we did return to x in k steps then $x \in C^k \subseteq L$. To show that we can return to any point in ri L in a finite number of steps we must demonstrate that C^∞ is convex. If x and y are in C^∞ then $x \in C^k$ and $y \in C^h$ for finite k and h. This implies that both x and y are in C^{kh}. Since C^{kh} is closed and convex, the line segment joining x and y is in $C^{kh} \subseteq C^\infty$. It follows Rockafellar [5], page 46 that ri $L = \text{ri}[\text{cl } C^\infty] = \text{ri } C^\infty \subseteq C^\infty$. For any $x \in$ ri L and $y \in S$, it is possible to move from y to x in a finite number of steps. This follows since the sequence of closed sets $R^t(y) \to L$, and $x \in$ ri L; thus there must exist a finite T such that $x \in R^T(y)$. In contrast, if $x \notin L$ and the initial $y \in L$, then it is not possible to move from y to x in a finite number of steps.

The next section presents a precise characterization of L in a special case. Proofs of Theorem (1) and Lemma (1) are presented in Section 5.

4. SPECIAL CASE - A CHARACTERIZATION OF L

This section examines the special case of

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{bmatrix}$$

where we assume

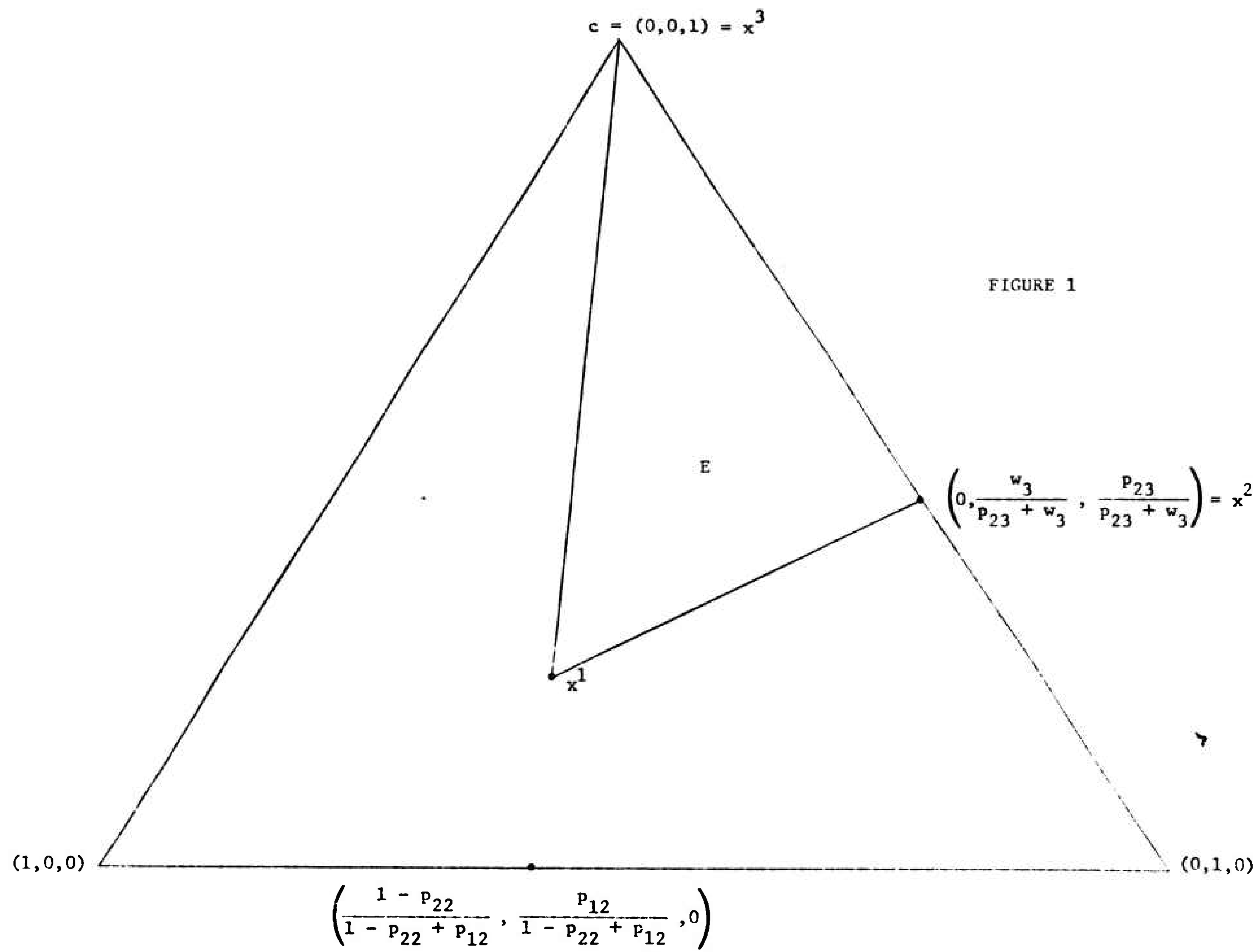
$$\text{(7)} \qquad \begin{aligned} &\text{(i)} \quad w_i > 0 \quad i = 1,2,3 \ . \\ &\text{(ii)} \quad p_{22} > p_{12} \\ &\text{(iii)} \quad w_2 \geq w_3 \ . \end{aligned}$$

The example corresponds to a three rank manpower hierarchy; e.g. assistant, associate, and full professors. Assumption (i) means it is possible to leave from any rank, (ii) is satisfied if $p_{ii} \geq 1/2$ for all i and (iii) indicates that withdrawal rates are higher in rank 2 than in rank 3. We shall present some numerical calculations below indicating that the main result of this section is true under more general conditions. The sets S and E are depicted in Figure 1. In this special case it is possible to obtain a precise characterization of the set L .

For $k = 1 , 2 , 3$ let $Q(k)$ be the matrix with

$$Q_{ij}(k) = \begin{cases} p_{ik} + w_i & \text{if } j = k \\ p_{ij} & \text{if } j \neq k \end{cases} .$$

In the manpower context, $Q(i)$ corresponds to making all new appointments in rank i . For $k = 1 , 2 , 3$ let x^k be the stationary vectors of the $Q(k)$; the x^k are proportional to the rows of $(I - P)^{-1}$ and are the extreme points of E .



c = (0,0,1) = x^3
E
$\left(0, \frac{w_3}{p_{23} + w_3}, \frac{p_{23}}{p_{23} + w_3}\right) = x^2$
x^1
(1,0,0)
(0,1,0)
$\left(\frac{1 - p_{22}}{1 - p_{22} + p_{12}}, \frac{p_{12}}{1 - p_{22} + p_{12}}, 0\right)$

FIGURE 1

Recall that C denotes convex hull. Define

$$F = C\{x^3Q^t(1), x^1Q^t(2), x^2Q^t(3) \quad t = 0,1,2, \ldots\} .$$

The sequence $x^iQ^t(j)$ simply starts at x^i and follows appointments in rank j only for all t. The points $x^2Q^t(3)$ for $t \geq 1$ all lie on the line segment $[x^2, x^3]$, thus

$$F = C\{x^3Q^t(1), x^1Q^t(2), x^2 \quad t = 0,1,2, \ldots\} .$$

Theorem 2:

Under the assumptions of this section,

$$L = F .$$

This theorem and the analysis developed in its proof have two corollaries.

Corollary 1:

For any cycle of length k, at least one element of the cycle lies in E.

Corollary 2:

For any solution of (1) and any $\varepsilon > 0$ neighborhood of E we have $x(t) \in N_\varepsilon(E)$ infinitely often.

The result allows us to make an excellent and easily calculated approximation of the set L and to gauge the effect of changing parameters on the set of limiting possibilities. Several cases are depicted below.

In Figure 2, we have $w_3 > w_2$ and Theorem 2 fails. However, in Figure 3, $w_3 > w_2$ and it is obvious that our approximation is valid. Thus it is sufficient but not necessary to have $w_3 \leq w_2$.

Figures 3 and 4 show two alternate P matrices and the effect of a change in the elements of P on both the equilibrium set E and the limiting set L.

$$P = \begin{bmatrix} .5 & .49 & 0 \\ 0 & .5 & .49 \\ 0 & 0 & .5 \end{bmatrix}$$

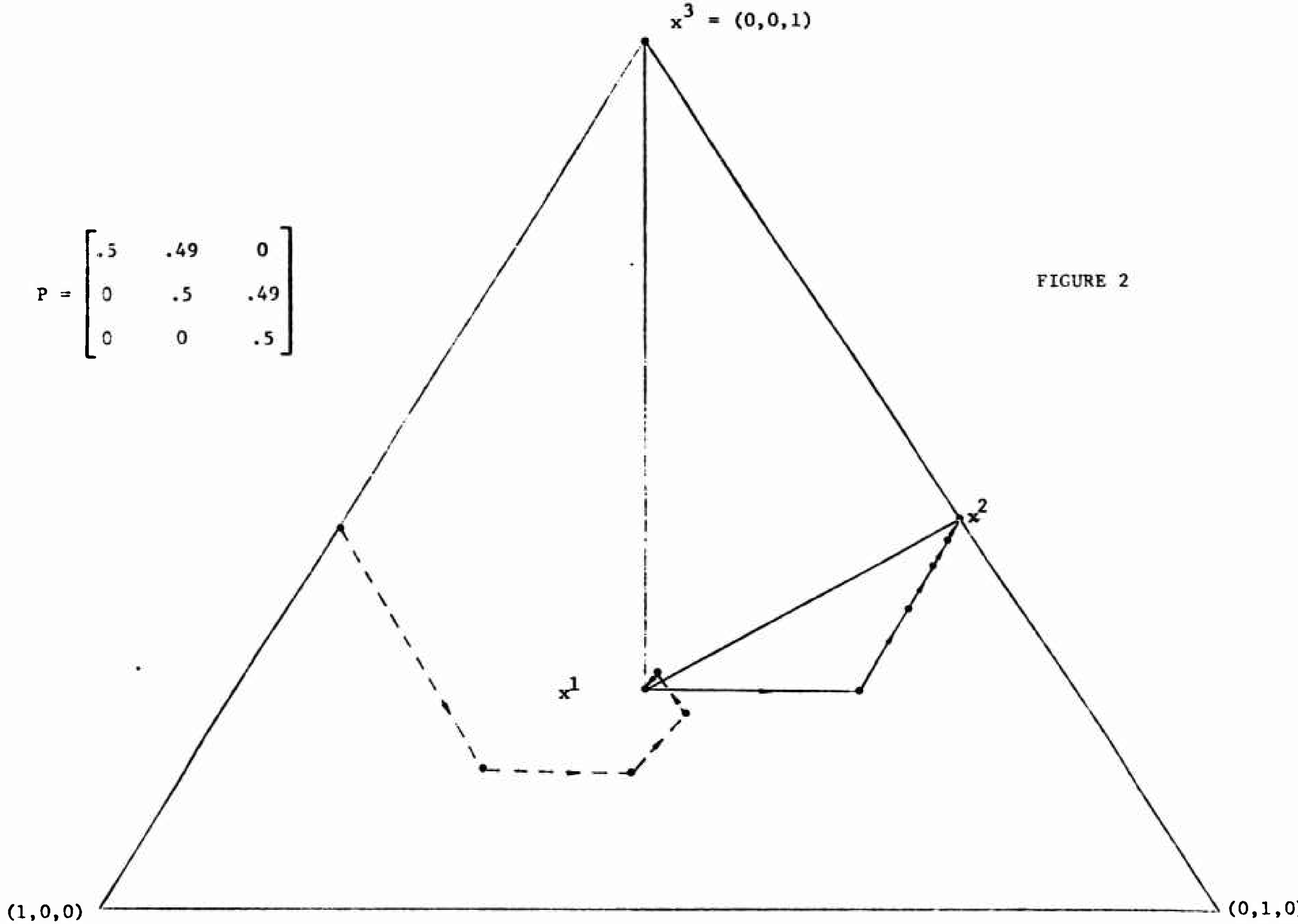


FIGURE 2

$$P = \begin{bmatrix} .75 & 0.1 & 0 \\ 0 & 0.85 & 0.1 \\ 0 & 0 & 0.9 \end{bmatrix}$$

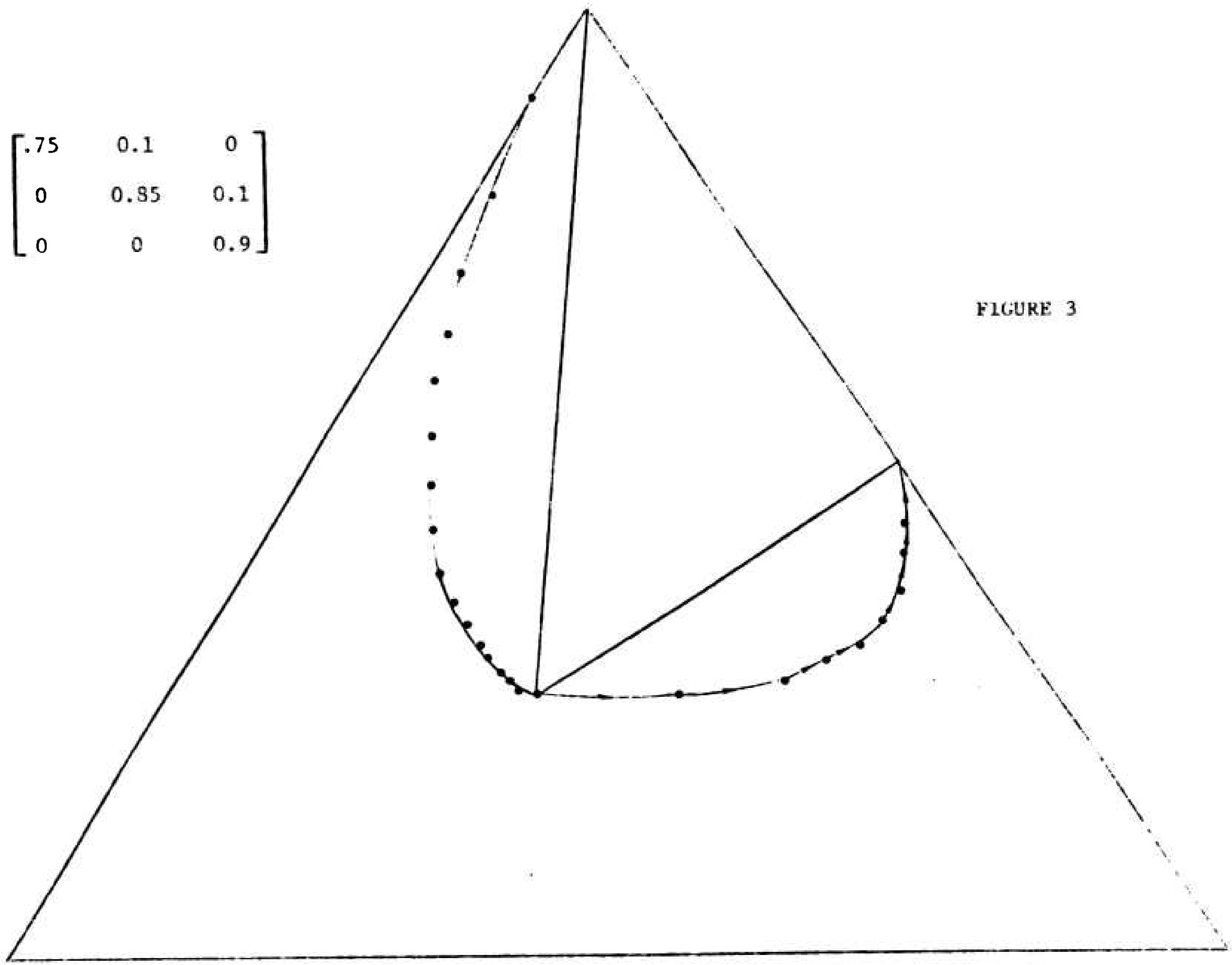


FIGURE 3

$$P = \begin{bmatrix} .8 & .15 & 0 \\ 0 & .75 & .15 \\ 0 & 0 & .9 \end{bmatrix}$$

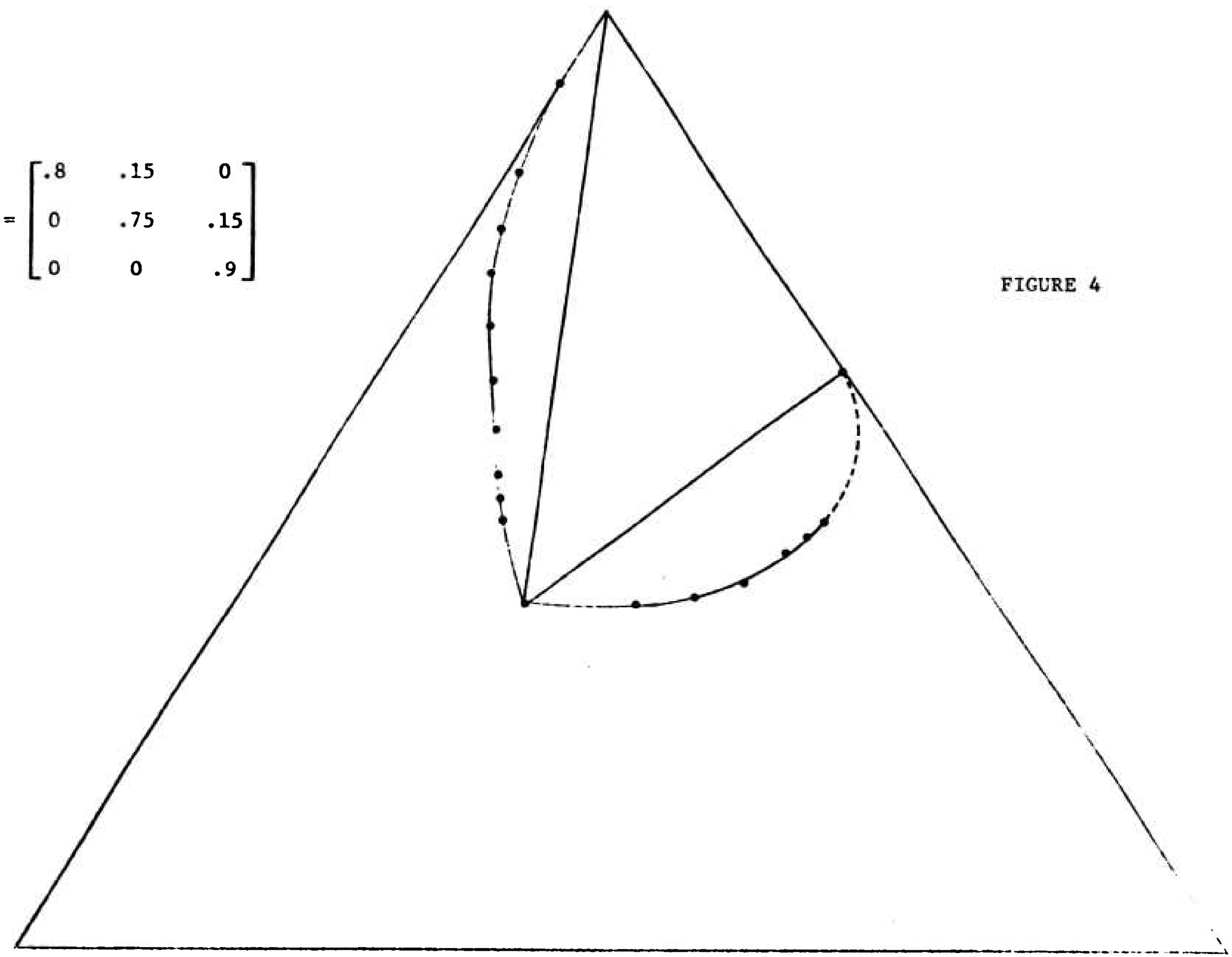


FIGURE 4

5. PROOFS

This section details the proofs of Lemma 1 and Theorems 1 and 2.

Proof of Lemma 4:

For the moment consider $z \in S$ fixed. If $x = y$, then $||x - y|| = 0$, and the Lemma is trivial. If $x \neq y$, let $v_i = x_i - y_i$ for all i and define

$$I^+ = \{i \mid v_i \geqq 0\} , \quad I^- = \{i \mid v_i < 0\} .$$

Note that:

$$\sum_{i=1}^{n} v_i = \sum_{I^+} v_i + \sum_{I^-} v_i = 0$$

and

$$||v|| = \sum_{I^+} v_i - \sum_{I^-} v_i$$

therefore

$$||v|| = 2 \sum_{I^+} v_i = -2 \sum_{I^-} v_i .$$

For each j let $u_j = \sum_{i=1}^{n} v_i Q_{ij}(z)$ and let $J^+ = \{j \mid u_j \geqq 0\}$, $J^- = \{j \mid u_j < 0\}$. Using the same logic as above,

$$||u|| = 2 \sum_{J^+} u_j = -2 \sum_{J^-} u_j .$$

In the first case,

$$||u|| = 2 \sum_{J^+} u_j = 2 \sum_{J^+} \left(\sum_{i=1}^{n} v_i Q_{ij}(z) \right)$$

$$= 2 \sum_{i=1}^{n} v_i \sum_{J^+} Q_{ij}(z) = 2 \sum_{i=1}^{n} v_i r_i(z) ,$$

where $$r_i(z) \triangleq \sum_{J^+} Q_{ij}(z) .$$

Hence

$$||u|| = 2\left(\sum_{I^+} v_i r_i(z) + \sum_{I^-} v_i r_i(z) \right) \leqq 2 \sum_{I^+} v_i r_i(z) .$$

If $h^+(z) \triangleq \operatorname*{Max}_{i\varepsilon I^+} [r_i(z)]$,

$$||u|| \leqq \left(2 \sum_{I^+} v_i \right) h^+(z) = h^+(z)||v|| .$$

In a similar fashion, if

$$s_i(z) \triangleq \sum_{J^-} Q_{ij}(z) \quad \text{and} \quad h^-(z) \triangleq \operatorname*{Max}_{i\varepsilon I^-} [s_i(z)] ,$$

then

$$||u|| = -2 \sum_{J^-} u_j = -2 \left[\sum_{I^+} v_i s_i(z) + \sum_{I^-} v_i s_i(z) \right] \leqq \left(-2 \sum_{I^-} v_i \right) h^-(z) .$$

Therefore $||u|| \leqq h^-(z)||v||$.

To summarize in terms of x , y , z ,

$$||(x - y)Q(z)|| \leqq ||x - y|| \operatorname{Min} \left[\operatorname*{Max}_{I^+} \left[\sum_{J^+} Q_{ij}(z) \right] , \operatorname*{Max}_{I^-} \left[\sum_{J^-} Q_{ij}(z) \right] \right] .$$

For each $Q_{ij}(z)$ we have $Q_{ij} \geq w^* z_j$ where $w^* = \operatorname{Min} \{w_i \mid i = 1,2, \ldots, n\} > 0$.

Thus

$$\underset{I^+}{\text{Min}} \left[\sum_{J^-} Q_{ij}(z) \right] \geq w^* \sum_{J^-} z_j$$

and

$$\underset{I^+}{\text{Max}} \left[\sum_{J^+} Q_{ij}(z) \right] \leq 1 - w^* \sum_{J^-} z_j \ .$$

Similarly

$$\underset{I^-}{\text{Max}} \left[\sum_{J^-} Q_{ij}(z) \right] \leq 1 - w^* \sum_{J^+} z_j \ .$$

It follows, that for all $z \in S$,

$$\text{Min} \left[\underset{I^+}{\text{Max}} \left[\sum_{J^+} Q_{ij}(z) \right], \underset{I^-}{\text{Max}} \left[\sum_{J^-} Q_{ij}(z) \right] \right] \leq 1 - \frac{w^*}{2} < 1 \ .$$

Thus $||(x - y)Q(z)|| \leq \left(1 - \frac{w^*}{2}\right)||x - y||$ ■

<u>Proof of Theorem (1)</u>:

First assume

(8) $$d[R(A),R(D)] \leq \sigma d(A,D)$$

for all closed nonempty sets A and D . It follows immediately that L is the unique fixed point of R , and with $D = L$ in (8), we obtain for each $t \geq 1$

$$d[R^t(A),L] \leq \sigma^t d[A,L] \ .$$

From Toole [7] we have $R^t(E) \subseteq \text{cl}\ C^\infty$ for all t , and $C^\infty \subseteq L$. Since $R^t(E)$ is an expanding sequence of closed sets converging to L , we must have $L = \text{cl}\ C^\infty$. Finally, for any nonempty A with $R(A) = A$ we have $\text{cl}\ R(A) = R(\text{cl}\ A) = \text{cl}\ A$, which implies $\text{cl}\ A = L$.

The proof is concluded by verifying Equation (8).

Let $h(u) = \underset{v\epsilon R(D)}{\mathrm{Min}} ||u - v||$. Then $\delta[R(A),R(D)] = \underset{u\epsilon R(A)}{\mathrm{Max}} h(u)$. Let u^* in $R(A)$ be such that

$$h(u^*) = \delta[R(A),R(D)] = \underset{v\epsilon R(D)}{\mathrm{Min}} ||u^* - v|| .$$

Note that

$$\delta[R(A),R(D)] = \delta[\{u^*\},R(D)] .$$

There exist some x in A and $z \epsilon S$ such that $u^* = xQ(z)$. Now choose $y \epsilon D$ so that $||x - y|| = \underset{u\epsilon D}{\mathrm{Min}}||x - y|| = \delta[\{x\},D]$. Therefore $yQ(z) \epsilon R(D)$ and

$$\delta[R(A),R(D)] = \underset{v\epsilon R(D)}{\mathrm{Min}} ||u^* - v|| \leqq ||(x - y)Q(z)|| .$$

From Lemma (1),

$$\delta[R(A),R(D)] \leqq \sigma||x - y|| = \sigma\delta[\{x\},D] .$$

Also

$$\delta[\{x\},D] \leqq \underset{u\epsilon A}{\mathrm{Max}}\ \delta[\{u\},D] = \delta[A,D] .$$

We have shown that

$$\delta[R(A),R(D)] \leqq \sigma\delta[A,D] .$$

It follows in a similar way that

$$\delta[R(D),R(A)] \leq \sigma\delta[D,A] .$$

Therefore

$$d[R(A),R(D)] \leq \sigma d(A,D) . \blacksquare$$

<u>Proof of Theorem (2):</u>

Since F is nonempty and closed, it suffices by Theorem 1, to prove that $R(F) = F$.

If $A \subseteq S$ is polyhedral with extreme points u^{ℓ} , $\ell = 1,2, \ldots,$ we may represent $R(A)$ as

$$C\{u^{\ell}Q(k) \quad \text{for} \quad \ell = 1,2, \ldots, \; k = 1,2,3\} .$$

Therefore

$$R(F) = C\{x^{i}Q^{n}((i + 1)\text{mod } 3)Q(j) \; , \; i = 1,2,3, \; j = 1,2,3, \; n = 0,1,2, \ldots\} .$$

It is clear that $R(F) \supseteq F$. In demonstrating $R(F) \subseteq F$, our attendant arguments will be sketchy--we shall omit a large amount of tedious algebra.

Let

$$V^2 = \{y : y \in S, y_2 < (yP)_2\}$$

and

$$V^3 = \{y : y \in S, y_3 < (yP)_3\} .$$

With the aid of Figure 5, we see that if $x^3Q^t(1) \in V^2 - V^3$, we must have

$$x^3Q^t(1)Q(3) \in C\{x^3, x^1Q(3), x^3Q^{t+1}(1)\}$$

and

$$x^3Q^t(1)Q(2) \in C\{x^3Q(2), x^1Q(2), x^3Q^{t+1}(1)\} .$$

Similarly, if $x^1Q^t(2)$ is in $V^3 - V^2$, then

$$x^1Q^t(2)Q(3) \in C\{x^2Q(3), x^1Q(3), x^1Q^{t+1}(2)\}$$

and

$$x^1Q^t(2)Q(1) \in C\{x^2Q(1), x^1, x^1Q^{t+1}(2)\} .$$

Also, it is clear that

$$x^2Q^t(3)Q(1) \in C\{x^3Q(1), x^2Q(1)\}$$

and

$$x^2Q^t(3)Q(2) \in C\{x^3, x^2\} .$$

Now it follows from the condition $p_{22} \geq p_{12}$ that $x^2Q(1) \in F$, and it is easy to show that we always have $x^1Q(3) \in F$. Hence we have established that each point of $R(F)$ is in F, provided that the sequences $\{x^3Q^t(1)\}$ and $\{x^1Q^t(2)\}$ remain in the sets $V^2 - V^3$ and $V^3 - V^2$ respectively.

The condition $p_{22} \geq p_{12}$ guarantees that if $x^1Q^t(2) \in V^3 - V^2$, then

(2) $$x^1Q^{t+1}(2) \in C\{x^1Q^t(2), x^2, (0,1,0)\} \subset V^3 - V^2 .$$

With the requirements $p_{22} \geq p_{12}$ and $w_2 \geq w_3$, we have that $x^3Q^t(1) \in V^2 - V^3$ implies

(3) $$x^3Q^{t+1}(1) \in C\{x^3Q^t(1), x^1, (1,0,0)\} \subset V^2 - V^3 ,$$

and the proof is complete.■

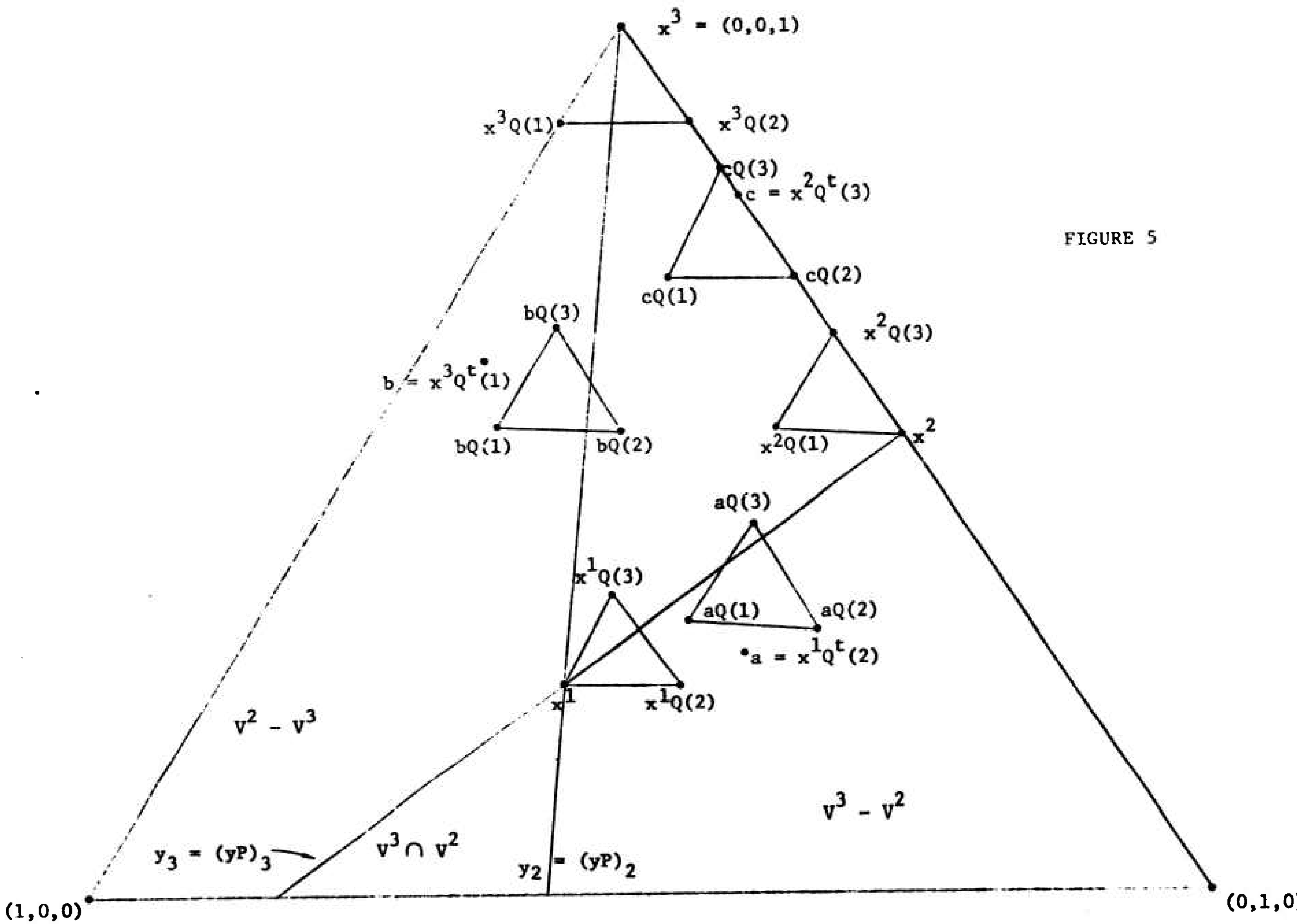

x^3 = (0,0,1)
x^3Q(1)
x^3Q(2)
cQ(3)
c = x^2Q^t(3)
cQ(1)
cQ(2)
x^2Q(3)
bQ(3)
b = x^3Q^t(1)
bQ(1)
bQ(2)
x^2Q(1)
x^2
aQ(3)
x^1Q(3)
aQ(1)
aQ(2)
a = x^1Q^t(2)
x^1
x^1Q(2)
V^2 - V^3
V^3 - V^2
y_3 = (yP)_3
V^3 ∩ V^2
y_2 = (yP)_2
(1,0,0)
(0,1,0)

FIGURE 5

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